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Electrostatic Field in Inhomogeneous Dielectric Media I. Indirect Boundary Element Method

NARENDA S. GOEL, FENGSHI GANG, AND ZHONGLIN KO

Department of Computer Science, Wayne State University, Detroit, Michigan 48202

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A computationally fast method is presented for calculating electrostatic field in arbitrary inhomogeneous dielectric media with open boundary condition. The method involves dividing the whole space into cubical cells and then finding effective dielectric parameters for interfacial cells consisting of several dielectrics. The electrostatic problem is then solved using either the indirect boundary element method described in this paper or the so-called volume element method described in the companion paper. Both methods are tested for accuracy by comparing the numerically calculated electrostatic fields against those analytically obtained for a dielectric sphere and dielectric ellipsoid in a uniform field and for a dielectric sphere in a point charge field. © 1995 Academic Press, Inc.

1. INTRODUCTION

The calculation of electrostatic potential and field for an arbitrary-shaped inhomogeneous dielectric medium with open boundary condition is an important problem in engineering, e.g., for calculating the capacitance of dielectric structures in connection with the design of electric devices and high voltage systems and for non-destructive testing of dielectric materials. There are several numerical techniques for calculating electrostatic fields which have been developed [1–5]. These techniques can be broadly divided into two categories:

1. The electrostatic problem is formulated in differential form and the differential equation is solved either by finite difference method (FDM) where the partial differential equation is replaced by a set of difference equations or by finite element method (FEM) where the complicated geometrical structures are divided into an irregular grid and the partial differential equation is solved by either a variational approach or a residual method. Both of these methods solve for the value of scalar electrostatic potential.

2. The electrostatic problem is formulated in an integral form. The moment method, charge simulative method (CSM) [4], and boundary element method (BEM) are typical methods which fall in this category. In the CSM, the continuous charge density on the surface of a dielectric is replaced by fictitious, discrete charges that are placed *inside* the dielectric and the values and locations of the charges are chosen to satisfy boundary conditions. In the BEM, sometimes also called the surface charge simulation method, one deals directly with charge distributed over the boundary.

The most desirable numerical method is the one which allows one to rapidly and accurately calculate electrostatic potential, electrostatic field, and other quantities of practical interest for an arbitrarily complex inhomogeneous dielectric media with open boundary conditions. One of the most commonly used methods is a hybrid of a finite element method and a boundary element method [2, 5]. In this scheme, BEM is applied for the extended nonboundary region. In the finite element part, one meshes the inhomogeneous region into irregular grid elements such that each element constitutes a homogeneous region. The remaining region is divided into a regular mesh and then one solves the appropriate equations for the given boundary conditions. Both the meshing and solving components are computationally intensive.

In this series of two papers, we present a novel, computationally fast and general approach for solving an electrostatic problem involving inhomogeneous dielectric media and open-boundary conditions. The approach consists of two main steps. In step 1, one tessellates the space of interest into cubical cells and calculates the effective dielectric constant for each cell. This method, known as the effective parameter for interfacial cells (EPIC) method, is described in Section 2. In step 2, one solves the electrostatic problem either by placing an appropriate charge distribution on the boundaries of cells or appropriate fictitious charges inside the cells. The first method, referred to as the indirect boundary element method (IBEM), is described in this paper, and the second method, referred to as the volume element method (VEM), is described in the following paper. Section 3 is devoted to the description and detail of IBEM. In Section 4, we give the results of applying the method for a dielectric sphere and a dielectric ellipsoid in a uniform electric field and for a dielectric sphere in the presence of an electric point charge, and compare the calculated electric field and other electric quantities with those obtained analytically.

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2. EFFECTIVE PARAMETER FOR INTERFACIAL CELLS (EPIC) METHOD

In this method, we tessellate the whole space into a regular cubical grid. In general, a cubical cell will consist of material of different dielectric properties. The key concept behind the method is to find an effective dielectric parameter for each cell such that the calculated physical effects are the same as for the original case of the cell being occupied by different materials. This approach was developed by us [7, 8] in connection with solving the heat conduction problem for inhomogeneous media involving objects of arbitrary shapes.

We describe EPIC by taking the case of a dielectric sphere (dielectric constant $\varepsilon = 5$) embedded in an infinite vacuum space. As shown in Fig. 1, we enclose the sphere in a 3 D rectangular parallelepiped (the cube as a special case) referred to as enclosure box. If one divides this box into a rectangular grid, say $5 \times 5 \times 5$, and takes any slice, say between $z = 1$ and $z = 2$, parallel to the x–y plane (see Fig. 1), then some cells or volume elements are totally occupied either by the spherical medium or by vacuum, but some are occupied by both spherical medium and vacuum. Let us refer to the mixed cells as *interfacial cells*. What we need is a method to determine effective dielectric parameters for such cells.

The procedure is illustrated for an interfacial cell, shown in Fig. 2, with two surfaces cutting through the cell and one surface totally enclosed in the cell. Thus, in this cell, there are regions m_1, m_2, m_3, and m_4 of four different dielectric properties. In general, an interfacial cell will have N regions, with anisotropic dielectric constants ε^i $(1 \le i \le N)$, with components ε_1^i, ε_2^i, ε_3^i in the x, y, and z directions. The EPIC method involves calculation of an effective dielectric constant ε (with components ε_1, ε_2, and ε_3) for the cell such that if the cell is occupied by one homogeneous material of dielectric constant ε, the physical effect will be the same as for the original cell occupied by several materials.

To determine ε_1, we divide the cell into 10×10 columns parallel to the x direction. For each column j $(1 \le j \le 100)$, in general, one will have K regions of lengths $L_1, \ldots, L_k$ of dielectric constants $\varepsilon_1, \ldots, \varepsilon_k$. The equivalent ε_1 for such a column of length L is calculated by using the concept of equivalent capacitor for K capacitors in series:

$$\varepsilon_1^j = \frac{L}{\sum_{i=1}^{K} (L_i/\varepsilon_1^i)}; \qquad L = \sum_{i=1}^{k} L_i \tag{1}$$

The overall ε_1 for the entire cell is then obtained by using the concept of equivalent capacitor for 100 capacitors in parallel:

$$\varepsilon_1 = \frac{\sum_{j=1}^{100} \varepsilon_1^j}{100}. \tag{2}$$

One follows a similar procedure for calculating ε_2 and ε_3 for y and z directions.

We have implemented a computer graphics method which takes its input, the shapes of various dielectric objects occupying the space and dielectric constants of various regions, and then gives the effective dielectric parameters for a specified level of tessellation. This method was developed in connection with the calculation of effective thermal conductivity and is described in detail in [9]. We should note that in the current implementation of this method, one requires about a minute of CPU time on a SUN SPARCstation 10 for a $20 \times 20 \times 20$ grid and only about 10 sec for a $10 \times 10 \times 10$ grid to calculate effective dielectric constants (compared to about 40 min for creating irregularly shaped elements for the finite element method). Further, the CPU time needed is independent of the complexity of the geometry of the dielectric objects.

We conclude this section by pointing out that even if all the dielectric media are isotropic, the interfacial cell can have an anisotropic dielectric constant ($\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3$) due to asymmetric distribution of different materials in the cell.

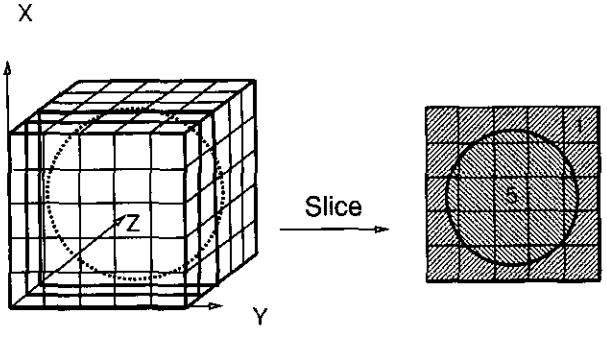

FIG. 1. *Dielectric sphere with $\varepsilon = 5$ embedded in vacuum.*

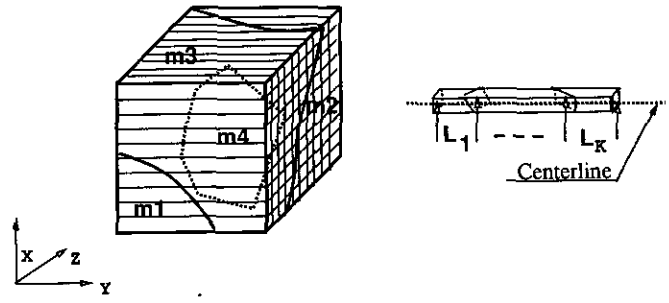

FIG. 2. Volume element subdividing and effective parameter calculation.

3. INDIRECT BOUNDARY ELEMENT METHOD

The EPIC method described in the preceding section allows one to replace the problem of an arbitrary shaped dielectric inhomogeneous media with $N \times N \times N$ cubical cells with

$S = (N + 1) \times (N + 1) \times (N + 1)$ surface elements, each of which is filled with one kind of material, with effective dielectric parameter specified for each material. We now describe the so-called indirect boundary element method which allows one to solve the electrostatic problem for such tessellated media.

The traditional boundary element method generally consists of the following steps [6, 10, 11]:

1. Transforming the governing ordinary differential equation into a boundary-integral equation.

2. Dividing the boundaries into a series of elements over which the source term is assumed to vary according to interpolation functions. The boundary integral equation is divided into a set of linear equations of boundary values at the nodes on the finite discrete boundary.

3. Obtaining a system of linear equations by imposing the prescribed boundary conditions of the problem, which are then solved to obtain the boundary values.

In the IBEM, we do not seek a direct solution of the potential; instead we obtain the polarized charge (source term) distributed on the boundary surface elements by forcing it to satisfy the boundary conditions.

In the following subsection we describe these steps in detail.

3.1. *Boundary Integral Equation—Single Cell*

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ define the dielectric constant in $x = x_1$, $y = x_2$, and $z = x_3$ directions for a cubical cell. The electrostatic potential Φ at a point $\vec{x} = (x_1, x_2, x_3)$ is given by the Laplace equation

$$\nabla_\varepsilon^2 \Phi = 0, \tag{3}$$

where

$$\nabla_\varepsilon^2 = \varepsilon_1 \frac{\partial^2}{\partial x_1^2} + \varepsilon_2 \frac{\partial^2}{\partial x_2^2} + \varepsilon_3 \frac{\partial^2}{\partial x_3^2}.$$

The Green's function (also called the fundamental solution) Φ^* which is the potential at the point $\vec{x}$ by the point source on the point $\vec{x}_i$ satisfies the equation

$$\nabla_\varepsilon^2 \Phi^* + \delta_i = 0, \tag{4}$$

where δ_i is the Dirac delta function

$$\delta_i = \delta(x_1 - x_{1i})\delta(x_2 - x_{2i})\delta(x_3 - x_{3i}).$$

Making the transformation

$$\xi_i = \frac{x_i}{\sqrt{\varepsilon_i}} \qquad (i = 1, 2, 3)$$

and using the property

$$\delta(x_1 - x_{1i}) = \delta(\sqrt{\varepsilon_1}(\xi_1 - \xi_{1i})) = \frac{1}{\sqrt{\varepsilon_1}}\delta(\xi_1 - \xi_{1i}),$$

the Green's function for the infinite region is given by

$$\Phi^*(i) = \frac{1}{4\pi r_0}\frac{1}{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}}, \tag{5}$$

where

$$r_0 = \sqrt{\frac{(x_1 - x_{1i})^2}{\varepsilon_1} + \frac{(x_2 - x_{2i})^2}{\varepsilon_2} + \frac{(x_3 - x_{3i})^2}{\varepsilon_3}}. \tag{6}$$

For our problem, let the polarized charged density at the boundary surface τ be $\sigma(\tau)$. The potential, therefore, is given by the integral equation

$$\Phi(x) = \int_\Gamma \sigma(\tau)\Phi^*(\tau, x)\, d\Gamma(\tau), \tag{7}$$

where Γ represents all the boundary surface.

On the boundary surfaces the boundary conditions are as follows:

$$\frac{\partial \Phi^{(1)}}{\partial n} - \frac{\partial \Phi^{(2)}}{\partial n} = \sigma(\tau)$$

$$\varepsilon_n^{(1)}\frac{\partial \Phi^{(1)}}{\partial n} - \varepsilon_n^{(2)}\frac{\partial \Phi^{(2)}}{\partial n} = 0, \tag{8}$$

where (1) stands for the inside of a surface and (2) for the outside.

Taking the derivative of Eq. (7) in the direction of the outward normal to the boundary surface and substituting the boundary conditions from Eq. (8), we have the boundary integral equation for the polarized charge density relating it to the normal derivative of the Green's Function Φ^* as

$$\sigma(x) = \frac{2(\varepsilon_n^{(1)} - \varepsilon_n^{(2)})}{(\varepsilon_n^{(1)} + \varepsilon_n^{(2)})}\left[E^0 + \int_\Gamma \sigma(\tau)\frac{\partial \Phi^*(\tau, x)}{\partial n}\, d\Gamma(\tau)\right], \tag{9}$$

where E^0 is the initial electric field. Equation (9) is the integral equation for one cubical cell in the presence of an electric field E^0. In the next section, we describe the modifications needed for the whole spacing consisting of many cubical cells containing different materials.

3.2. *Boundary Integral Equation—Many Cells*

To obtain the boundary integral equation for the whole space, we discretize the charge density at the surface (a square) of each cell. For each square element, the surface charge density

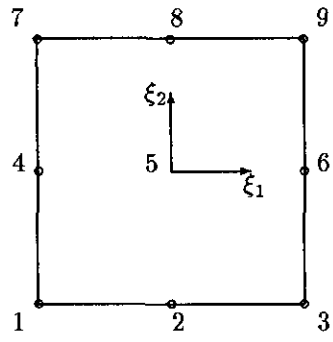

FIG. 3. Nine-noded Lagrangian quadrilateral element.

can be described by the densities at 9 nodes, as shown in Fig. 3. This approximation for the boundary element is called a Lagrangian quadrilateral element [6]. For the boundary element i, let σ_{ik} be the charge density at the node k ($k = 1, ..., 9$). Therefore, the charge density at a point (ξ_1, ξ_2) on the surface of element i, defined by dimensionless coordinates (ξ_1, ξ_2), with ξ_1 and ξ_2 varying from -1 to $+1$, is given by [6]

$$\begin{aligned}\sigma_i(\xi_1, \xi_2) &= \frac{1}{4}\xi_1(\xi_1 - 1)\xi_2(\xi_2 - 1)\sigma_{i1} + \frac{1}{2}(1 - \xi_1^2)\xi_2(\xi_2 - 1)\sigma_{i2} \\ &\quad + \frac{1}{4}\xi_1(\xi_1 + 1)\xi_2(\xi_2 - 1)\sigma_{i3} + \frac{1}{2}\xi_1(\xi_1 - 1)(1 - \xi_2^2)\sigma_{i4} \\ &\quad + (1 - \xi_1^2)(1 - \xi_2^2)\sigma_{i5} + \frac{1}{2}\xi_1(\xi_1 + 1)(1 - \xi_2^2)\sigma_{i6} \\ &\quad + \frac{1}{4}\xi_1(\xi_1 - 1)\xi_2(\xi_2 + 1)\sigma_{i7} + \frac{1}{2}(1 - \xi_1^2)\xi_2(\xi_2 + 1)\sigma_{i8} \\ &\quad + \frac{1}{4}\xi_1(\xi_1 + 1)\xi_2(\xi_2 + 1)\sigma_{i9} \\ &= \sum_{k=1}^{9} A_k\sigma_{ik}. \end{aligned} \tag{10}$$

To calculate the surface charge density for the collection of cells, we need to include the interaction between various surface elements and consider the whole boundary surface which consists of the total grid surfaces (boundary elements) of the interested space. We renumber the $S = (N + 1) \times (N + 1) \times (N + 1)$ surface elements as 1, 2, ..., S. By substituting Eq. (10) into Eq. (9), for each surface element i and its node k we have the discretized integral equation for the charge density σ_{ik},

$$\sigma_{ik} = c_i\left[E_{ik}^0 + \sum_{j=1, j\neq i}^{S}\int_{\Gamma_j}\sigma_j\frac{\partial\Phi_{jik}^*}{\partial n}d\Gamma_j\right], \tag{11}$$

where c_i is a constant for each element i,

$$c_i = \frac{2(\varepsilon_n^{(1)} - \varepsilon_n^{(2)})}{(\varepsilon_n^{(1)} + \varepsilon_n^{(2)})}, \tag{12}$$

E_{ik}^0 is the external electrical field at node k of the element i, j is an index for all surface elements, and Φ_{jik}^* is the Green's function at the node k of element i with the charge density σ_j at element j. Substituting Eqs. (10) and (5) into Eq. (11), we get the linear discrete equations

$$E_{i,k}^0 = \sum_{j=1}^{S}\sum_{m=1}^{9}\sigma_{j,m}H_{jm,ik} \tag{13}$$

where $H_{jm,ik}$ is an "interaction integral" between node k of element i and node m of element j. For $j \neq i$,

$$H_{jm,ik} = -\int_{\Gamma_j}A_k\frac{\partial\Phi_{jik}^*}{\partial n}d\Gamma_j \tag{14a}$$

where A_k is defined by Eq. (10).

For $j = i$, $m = k$, i.e., for interaction between a node with itself,

$$H_{im,im} = c_i^{-1}, \tag{14b}$$

while for $j = i$, $m \neq k$, i.e., for interaction between two different nodes on the same element,

$$H_{im,ik} = 0, \qquad m \neq k. \tag{14c}$$

As shown in the Appendix, the integral in Eq. (14a) can be calculated analytically. Renumbering the $9 \times S = M$ nodes as 1, ..., M, Eq. (13) can be written as a matrix equation

$$\mathbf{H}\sigma = \mathbf{Q}, \tag{15a}$$

i.e.,

$$\begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,M-1} & h_{1,M} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,M-1} & h_{2,M} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{M,1} & h_{M,2} & \cdots & h_{M,M-1} & h_{M,M} \end{pmatrix}\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_M \end{pmatrix} = \begin{pmatrix} E_1^0 \\ E_2^0 \\ \vdots \\ E_M^0 \end{pmatrix}. \tag{15b}$$

This equation needs to be solved for the $M = 9 \times S$ unknown variables $\sigma_1, \sigma_2, ..., \sigma_M$ which can then be used to obtain electrostatic potential at an arbitrary point.

3.3. *Solution of Matrix Equation*

The matrix in Eq. (15) is a dense, asymmetric, and diagonal dominate matrix. Therefore, it can be solved by using the stan-

dard Gauss–Seidel method (using the overrelaxation version of this method, the convergence deteriorates). We note that one can reduce the size of the matrix by eliminating non-boundary cells, i.e., those cells for which the coefficient c_i, defined by Eq. (12), is equal to zero so that there is no boundary surface charge. For a specific case of a $10 \times 10 \times 10$ grid, the total number of the boundary element is $11 \times 11 \times 11 \times 3 = 3993$. If one eliminates the non-boundary elements, one is left with 834 interfacial elements, reducing the size of the matrix H in Eq. (15) from 35937 ($= 9 \times 3993$) $\times$ 35937 to 7506 ($= 9 \times 834$) $\times$ 7506.

After solving the matrix Eq. (15) for the surface charge densities at nine nodes of each surface element, one can obtain the impending electrostatic field at any given point by using a discrete integral method. By adding the impending field and the initial field, one can obtain the total final field. One can also obtain other electrostatic quantities of interest like the total charge and dipole moment. In the next section, we describe the implementation of the procedure and give the results of its application for sample problems.

4. IMPLEMENTATION, EXPERIMENTS, AND RESULTS

To recap, the indirect boundary element method described in this paper consists of the following steps:

1. Take as input the geometrical shapes of different dielectric objects, together with their dielectric properties.

2. Divide the space into a cubical grid and use a computer-graphics-based algorithm version of EPIC to obtain effective dielectric constants for each of the cells.

3. Calculate the coefficient c_i to determine the boundary surface elements, calculate the relative coefficient $H_{i,j}$ between boundary surface element i and element j, and calculate the normal components of the initial electric field on every boundary surface element.

4. Solve the matrix Eq. (15) to obtain the surface charge distributions on the boundary surface elements.

5. Calculate the final field at any desired position and other parameters of interest like the total positive charge, dipole vector, capacitance and inductance.

We wrote a computer program in C language and implemented the method on a SUN SPARCstation 10. We tested our numerical solution against analytical results for three cases: (1) a dielectric sphere in a uniform parallel field; (2) an ellipsoid in a uniform parallel field, and (3) a dielectric sphere in a point charge field. For each of these cases, we treated the conducting material as a special case of dielectric material with very high dielectric constant ($\varepsilon = 10^5$).

4.1. *Dielectric Sphere in a Parallel Field*

Here we have a dielectric sphere of radius 0.5, embedded in a uniform electric field of strength 100, parallel to the z-direction. The analytical solution for this problem is well known.

In Fig. 4 are given the angular charge distribution on the surface of the sphere as charge density vs. angle in radian (angle = 0 corresponds to the direction parallel to the field, and angle = $\pi/2$ to the direction perpendicular to the field) for three resolution levels of gridding, $5 \times 5 \times 5$, $10 \times 10 \times 10$, and $20 \times 20 \times 20$. As to be expected, the resolution level of

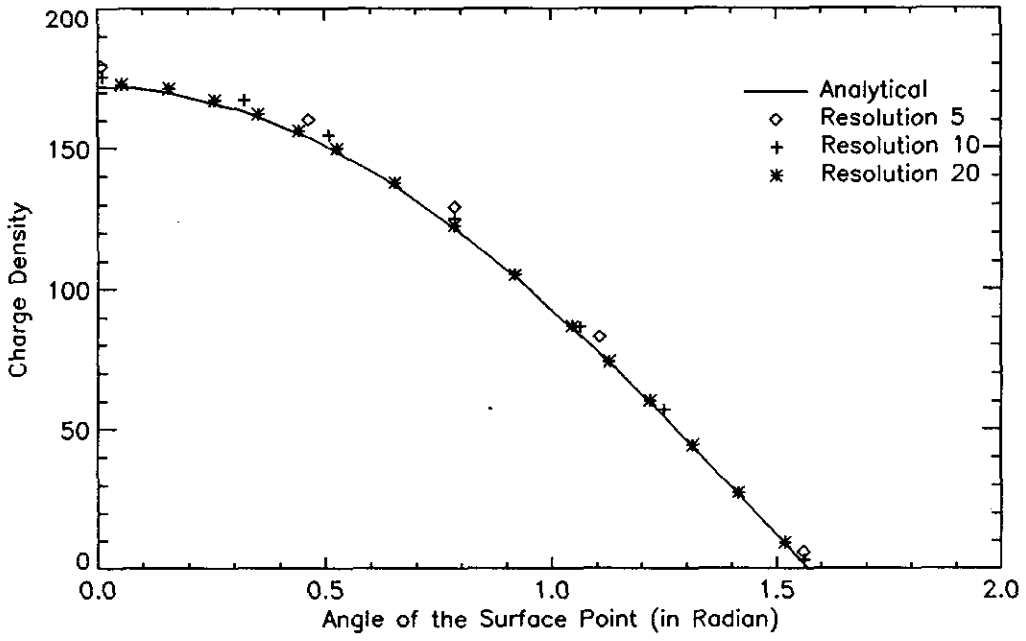

FIG. 4. Charge distribution on the sphere between 0 to $\pi/2$.

TABLE I

Comparison of Numerically Calculated and Analytical Results for the Dipole Moment (in the z-Direction) and Total Positive Charge for a Dielectric Sphere in a Uniform Electric Field of Strength 100 in the z-Direction

Dielectric constant ε	Dipole moment			Total positive charge		
	Calculated	Analytical	% Difference	Calculated	Analytical	% Difference
5	89.85	89.76	0.10	134.75	134.64	0.08
25	140.37	139.63	0.53	209.92	209.44	0.2
10^5	158.68	157.08	1.02	237.78	235.61	0.92

$20 \times 20 \times 20$ gives the most accurate result, the resolution level of $10 \times 10 \times 10$ gives a somewhat less accurate but still acceptable result, and the resolution level of $5 \times 5 \times 5$ gives a very poor estimate. The computer times for these three levels of resolution are 60 min, 60 sec, and 15 sec, respectively. We find a resolution level of $10 \times 10 \times 10$ as an optimum choice keeping accuracy and computer time in mind. We carried out the remaining simulations for this level of resolution.

In Table I is given the comparison between numerical and analytical results for the dipole moment in the z-direction (dipole moments in the x and y directions are zero for both numerical and analytical cases) and the total positive charge. The numerical method is very accurate with error less than or equal to 0.7% for a large range of dielectrics ($\varepsilon \leq 1000$), and for a near-conducting sphere the error is around 1%.

4.2. *Dielectric Ellipsoid in a Parallel Field*

In this case, the dielectric sphere is replaced by a dielectric ellipsoid of semi-axes 3, 4, and 5, with its surface defined by the equation

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z^2}{5^2} = 1$$

with initial field of strength 10 parallel to the z-axis.

Because of the relative sizes of the three axes, we divided the space into a grid of $6 \times 8 \times 10$. In Table II are given the results for the dipole moment in the z-direction and the impending field inside the ellipsoid (we omit the comparison for the total positive charge as we did for the sphere case because the value for the analytical case is not available). As one can see, the numerical values are quite accurate, even for the near-conducting sphere.

4.3. *Dielectric Sphere in a Point Charge Field*

This case is similar to the first case except that the initial field is caused by a point charge of value 2000 located at a distance d ($d > 0.5$) from the center of the sphere along the z-axis.

When the point charge is very far from the sphere, the initial field near the sphere is almost a parallel field, and, therefore, our method should yield accurate results. This is shown in Table III where we give the result for $d = 3$. As the charge is brought near the sphere, the accuracy of the method decreases. In Table IV we give the results for the worst case of near-conducting sphere for various values of the distance d. The method has limited accuracy for $d = 0.6$. When $d = 0.5$, i.e., when the point source is on one of the boundary element, we have a singularity when we solve the integral equations and the method fails. We have not found a way to overcome this problem.

TABLE II

Comparison of Numerically Calculated and Analytical Results for the Dipole Moment (in the z-Direction) and Impending Field in the z-Direction Inside a Dielectric Ellipsoid of Semi-axes 3, 4, and 5, Kept in a Uniform Electric Field of Strength 10 in the z-Direction

Dielectric constant ε	Dipole moment			Inside impending field		
	Calculated	Analytical	% Difference	Calculated	Analytical	% Difference
5	5154.85	5129.2	0.15	−4.91	−4.90	0.20
25	8973.17	8923.2	0.56	−8.58	−8.52	0.67
10^5	10600.8	10472.0	1.23	−10.15	−10.0	1.54

TABLE III

Comparison of Numerically Calculated and Analytical Results for the Dipole Moment (in the z-Direction) for a Dielectric Sphere of Radius 0.5 in an Electric Field Caused by a Point Charge of Value 2000 Located at a Distance 3 from the Center of the Sphere

Dielectric constant ε	Dipole moment		
	Calculated	Analytical	% Difference
5	−15.90	−15.87	0.20
25	−24.90	−24.69	0.86
10^5	−28.31	−27.78	1.92

APPENDIX: CALCULATION OF INTERACTION INTEGRAL $H_{jm,ik}$

In this appendix, we show that the integrals $H_{jm,ik}$ defined by Eq. (14a) can be numerically integrated.

For $m = 1$, $j \neq i$ (i.e., node 1 on the jth element), and if the distance between node m and node k (of element i) is not zero, from Eqs. (10) and (14a)

$$H_{j1,ik} = \frac{l_1 l_2}{16} \int_{\Gamma_j} \xi_1(\xi_1 - 1)\xi_2(\xi_2 - 1) \frac{\partial \Phi^*_{jik}}{\partial n} d\Gamma_j$$

$$= \frac{l_1 l_2}{16} \frac{\partial \Phi^*_{jik}}{\partial n} \int_{-1}^{1} d\xi_2 \int_{-1}^{1} d\xi_1 \xi_1(\xi_1 - 1)\xi_2(\xi_2 - 1)$$

$$= \frac{l_1 l_2}{36} \frac{\partial \Phi^*_{jik}}{\partial n}, \tag{A1}$$

where l_i, $i = 1, 2, 3$, are the length of a cell in x, y, and z directions. For a cubical cell, $l_1 = l_2 = l_3$. The normal derivative of Φ^*_{jik} can be expressed as

$$\frac{\partial \Phi^*_{jik}}{\partial n} = \frac{(x_{n,j1} - x_{n,ik})}{((x_{1,j1} - x_{1,ik})^2 + (x_{2,j1} - x_{2,ik})^2 + (x_{3,j1} - x_{3,ik})^2)^{3/2}}, \tag{A2}$$

where $x_{ik}n$ ($n = 1, 2, 3$) stands for the normal coordinate of node k on element i. Substituting Eq. (A2) into Eq. (A1), we get

$$H_{j1,ik} = \frac{l_1 l_2}{36} \frac{(x_{n,j1} - x_{n,ik})}{((x_{1,j1} - x_{1,ik})^2 + (x_{2,j1} - x_{2,ik})^2 + (x_{3,j1} - x_{3,ik})^2)^{3/2}}, \tag{A3}$$

Repeating the procedure for other nodes $m = 2, 3, \ldots, 9$ on the jth element, one obtains the general equation for $m = 1, 2, \ldots, 9$,

$$H_{j1,ik} = \frac{l_1 l_2}{a} \frac{(x_{n,jm} - x_{n,ik})}{((x_{1,jm} - x_{1,ik})^2 + (x_{2,jm} - x_{2,ik})^2 + (x_{3,jm} - x_{3,ik})^2)^{3/2}}, \tag{A4}$$

where $a = 9$ for $m = 2, 4, 5, 6, 8$ and $a = 36$ for $m = 1, 3, 7, 9$.

We should make an important observation. If $l_1 = l_2 = l_3$, i.e., we divide the whole space into a cubical grid and fix the number of grids to say $N \times N \times N$, then the matrix element $H_{jm,ik}$, defined by Eq. (A4), is effectively independent of the dielectric properties of the media and hence needs to be calculated only once. Restating another way, we can a priori calculate and store the elements of the matrix H for a given fineness of the cubical gridding process and thus make the indirect boundary element method computationally very fast as we solve different electrostatic problems.

TABLE IV

As Table III Except that the Dielectric Constant of the Sphere Is Kept Fixed at 10^5 and the Distance of the Charge Is Varied from 0.6 to 6

Distance of charge	Dipole moment		
	Calculated	Analytical	% Difference
0.6	−740.97	−694.44	6.70
1.0	−258.03	−250.00	3.21
3	−28.31	−27.78	1.92
6	−7.02	−6.94	1.12

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